Verma modules over a Block Lie algebra

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Abstract. Let $\mathcal{B}$ be the Lie algebra with basis $\{L_{i,j},\,C\,|\,i,j\in\mathbb{Z}\}$ and relations $[L_{i,j},L_{k,l}]=((j+1)k-i(l+1))L_{i+k,j+l}+i\delta_{i,-k}\delta_{j+l,-2}C$, $[C,L_{i,j}]=0$. It is proved that an irreducible highest weight $\mathcal{B}$-module is quasifinite if and only if it is a proper quotient of a Verma module. For an additive subgroup Γ of $\mathbb{F}$, there corresponds to a Lie algebra $\mathcal{B}(\Gamma)$ of Block type. Given a total order $\succ$ on Γ and a weight Λ, a Verma $\mathcal{B}(\Gamma)$-module $M(\Lambda,\succ)$ is defined. The irreducibility of $M(\Lambda,\succ)$ is completely determined.

Key words: Verma modules, Lie algebras of Block type, irreducbility, quasifinite modules
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§1. Introduction

Since a class of infinite dimensional simple Lie algebras was introduced by Block [B], generalizations of Block algebras have been studied by some authors, partially because they are closely related to the Virasoro algebra or algebras associated quantum plane (e.g., [DZ, LW, OZ, S1–S4, SZ, X1, X2, ZM]).

For an additive subgroup Γ of a field $\mathbb{F}$ of characteristic 0, we consider in this paper the *Lie algebra* $\mathcal{B}(\Gamma)$ *of Block type* with basis $\{L_{\alpha,i},C\,|\,\alpha\in\Gamma,\ i\in\mathbb{Z}\}$ and relations

$$\begin{aligned}&[L_{\alpha,i},L_{\beta,j}]=((i+1)\beta-(j+1)\alpha)L_{\alpha+\beta,i+j}+\alpha\delta_{\alpha,-\beta}\delta_{i+j,-2}C,\\&[C,L_{\alpha,i}]=0.\end{aligned}\tag{1.1}$$

This particular Block type Lie algebra attracts our attention because it is also a special case of Cartan type S Lie algebras (e.g., [X, SX1]), or Poisson (or Hamiltonian) algebras (e.g., [X, S6, SX2]). One of our motivations to study representations of this Lie algebra is to better understand representations of Lie algebras of 4 families of Cartan type.

The Lie algebra $\mathcal{B}(\Gamma)$ is Γ-graded (but not finitely graded):

$$\mathcal{B}(\Gamma)=\bigoplus_{\alpha\in\Gamma}\mathcal{B}(\Gamma)_\alpha,\quad\text{where}\ \ \mathcal{B}(\Gamma)_\alpha=\mathrm{span}\{L_{\alpha,i}|\ i\in\mathbb{Z}\}+\delta_{\alpha,0}\mathbb{F}C.\tag{1.2}$$

Note that $\mathcal{B}(\Gamma)_0$ is an infinite dimensional commutative subalgebra of $\mathcal{B}(\Gamma)$ (but it is not a Cartan subalgebra). For a total order "$\succ$" on Γ compatible with its group structure (in case $\Gamma=\mathbb{Z}$, we always choose the normal order on $\mathbb{Z}$), we denote $\Gamma_\pm=\{x\in\Gamma\,|\,\pm x\succ 0\}$. Then $\Gamma=\Gamma_+\cup\{0\}\cup\Gamma_-$, and we have the *triangular decomposition*

$$\mathcal{B}(\Gamma)=\mathcal{B}(\Gamma)_-\oplus\mathcal{B}(\Gamma)_0\oplus\mathcal{B}(\Gamma)_+,\quad\text{where}\ \ \mathcal{B}(\Gamma)_\pm=\bigoplus_{\pm\alpha\succ 0}\mathcal{B}(\Gamma)_\alpha.$$

Thus we have the notion of a Verma module $M(\Lambda,\succ)$ with respect to a function, called a *weight*, $\Lambda\in\mathcal{B}(\Gamma)_0^*$ (the dual space of $\mathcal{B}(\Gamma)_0$) and the order "$\succ$" (cf. (2.1)).

A weight Λ is described by the *central charge* $c=\Lambda(C)$ and its *labels* $\Lambda_i=\Lambda(L_{0,i-1})$ for $i\in\mathbb{Z}$. For $j\in\mathbb{Z}$, we introduce the *j-th generating series*

$$\Delta_\Lambda^{(j)}(z)=\sum_{i=0}^{\infty}\frac{\Lambda_{i+j}}{i!}z^i. \tag{1.3}$$

A function $\Delta(z)$ is called a *quasipolynomial* if it is a linear combination of functions of the form $p(z)e^{\alpha z}$, where $p(z)\in\mathbb{F}[z]$, $\alpha\in\mathbb{F}$. Recall [KL, KR] the well-known characterization that a formal power series is a quasipolynomial if and only if it satisfies a nontrivial linear differential equation with constant coefficients.

A $\mathcal{B}(\Gamma)$-module V is *quasifinite* if V is finitely Γ-graded, namely,

$$V=\bigoplus_{\alpha\in\Gamma}V_\alpha\quad\text{with}\quad\mathcal{B}(\Gamma)_\alpha V_\beta\subset V_{\alpha+\beta},\ \ \dim V_\alpha<\infty\quad\text{for}\quad\alpha,\beta\in\Gamma.$$

Quasifinite modules are closely studied by some authors, e.g., [KL, KR, KWY, LZ, S3, S4]. Quasifinite irreducible modules over the Lie algebras $\mathcal{B}'(\Gamma)$ are proved to be a highest or lowest weight module in [S5], where

$$\mathcal{B}'(\Gamma)=\mathrm{span}\{L_{\alpha,i},C\,|\,\alpha\in\Gamma,\,i\geq-1\} \tag{1.4}$$

is a subalgebra of $\mathcal{B}(\Gamma)$.

Since each grading space $\mathcal{B}(\Gamma)_\alpha$ in (1.2) is infinite-dimensional, the classification of quasifinite modules is a nontrivial problem as pointed in [KL, KR]. Thus the classification of graded modules with infinite dimensional grading spaces is a nontrivial problem as well.

The main results in this paper are the following.

Theorem 1.1 *Suppose V is an irreducible highest weight $\mathcal{B}(\mathbb{Z})$-module of weight Λ. The following statements are equivalent:*

(1) *V is quasifinite;*

(2) *V is a proper quotient of the Verma module $M(\Lambda,\succ)$;*

(3) *$\Sigma_\Lambda^{(j)}(z):=z\Delta_\Lambda^{(-j)}(z)-j\Delta_\Lambda^{(-j-1)}(z)-\frac{c}{j!}z^j$ is a quasipolynomial and satisfies the same nontrivial linear differential equation with constant coefficients for all $j\in\mathbb{Z}$.*

Theorem 1.2 *Let $M(\Lambda,\succ)$ be the Verma $\mathcal{B}(\Gamma)$-module with highest weight Λ (cf. (2.1)).*

(1) *With respect to a dense order "$\succ$" of* Γ (*cf.* (2.8)),

$$M(\Lambda,\succ) \text{ is irreducible} \quad \Longleftrightarrow \quad \Lambda \neq 0.$$

Moreover, in case $\Lambda = 0$, *the submodule*

$$M'(0,\succ) = \textstyle\sum_{k>0,\,\alpha_1,\cdots,\alpha_k\in G_+} \mathbb{F}L_{-\alpha_1,i_1}\cdots L_{-\alpha_k,i_k}v_0 \quad \text{is irreducible}$$

$$\Longleftrightarrow \quad \forall\, x,y\in\Gamma_+,\ \exists\, n\in\mathbb{Z}_+ \ \text{ such that } \ nx\succ y.$$

(2) *With respect to a discrete order "$\succ$"* (*cf.* (2.9)),

$$M(\Lambda,\succ) \text{ is irreducible} \quad \Longleftrightarrow \quad \mathcal{B}(a\mathbb{Z})\text{-module } M_a(\Lambda,\succ) \text{ is irreducible.}$$

§2. Proof of the main result

Let Γ be any additive subgroup of $\mathbb{F}$. Denote by $U = U(\mathcal{B}(\Gamma))$ the universal enveloping algebra of $\mathcal{B}(\Gamma)$. For any $\Lambda \in \mathcal{B}(\Gamma)_0^*$, let $I(\Lambda,\succ)$ be the left ideal of U generated by the elements

$$\{L_{\alpha,i} \mid \alpha\succ 0, i\in\mathbb{Z}\}\cup\{h-\Lambda(h)\cdot 1 \mid h\in\mathcal{B}(\Gamma)_0\}.$$

Then the *Verma* $\mathcal{B}(\Gamma)$*-module* with respect to the order "$\succ$" is defined as

$$M(\Lambda,\succ) = U/I(\Lambda,\succ), \tag{2.1}$$

which has a basis consisting of all vectors of the form

$$\begin{aligned} L_{-\alpha_1,i_1}L_{-\alpha_2,i_2}\cdots L_{-\alpha_k,i_k}v_\Lambda, \quad &\text{where} \quad i_j\in\mathbb{Z},\ 0\prec\alpha_1\preccurlyeq\cdots\preccurlyeq\alpha_k,\\ &\text{and} \qquad i_s\le i_{s+1} \text{ if } \alpha_s=\alpha_{s+1}. \end{aligned} \tag{2.2}$$

where v_Λ is the coset of 1 in $M(\Lambda,\succ)$. Thus $M(\Lambda,\succ)$ is a *highest weight* $\mathcal{B}(\Gamma)$*-module* in the sense

$$M(\Lambda,\succ) = \oplus_{\alpha\preccurlyeq 0}M_\alpha, \tag{2.3}$$

where $M_0 = \mathbb{F}v_\Lambda$, and M_α for $\alpha\prec 0$ is spanned by vectors in (2.2) with $\alpha_1+\cdots+\alpha_k=-\alpha$. So it is a Γ-graded $\mathcal{B}(\Gamma)$-module with

$$\dim M_{-\alpha} = \infty \quad \text{for} \quad \alpha\in\Gamma_+. \tag{2.4}$$

For any $a\in\Gamma$, we denote

$$\mathcal{B}(a\mathbb{Z}) = \text{span}\{L_{na,k}, C \mid n,k\in\mathbb{Z}\}, \tag{2.5}$$

which is a subalgebra of $\mathcal{B}(\Gamma)$ isomorphic to $\mathcal{B}(\mathbb{Z})$. We also denote

$$M_a(\Lambda,\succ) \quad = \quad \mathcal{B}(a\mathbb{Z})\text{-submodule of } M(\Lambda,\succ) \text{ generated by } v_\Lambda. \tag{2.6}$$

Denote

$$B(\alpha)=\{\beta\in\Gamma\mid 0\prec\beta\prec\alpha\}\quad\text{for}\ \ \alpha\in\Gamma_+. \tag{2.7}$$

The order " $\succ$" is called *dense* if

$$\#B(\alpha)=\infty\quad\text{for all}\ \ \alpha\in\Gamma_+, \tag{2.8}$$

it is *discrete* if

$$B(a)=\emptyset\quad\text{for some}\ \ a\in\Gamma_+. \tag{2.9}$$

Theorem 1.2 is an analog of a result in [WS] (where the Lie algebras $\mathcal{B}'(\Gamma)$ were considered instead of $\mathcal{B}(\Gamma)$). Since the proof is similar to that in [WS], we omit the detail.

Now suppose $\Gamma=\mathbb{Z}$, and from now on, we shall only consider the Lie algebra

$$\mathcal{B}=\mathcal{B}(\mathbb{Z}). \tag{2.10}$$

The Lie algebra $\mathcal{B}$ has a nice realization in the space $\mathbb{F}[x^{\pm1},t^{\pm1}]\oplus\mathbb{F}C$ by setting

$$L_{i,j}=x^it^{j+1}\quad\text{for any}\ i,j\in\mathbb{Z},$$

with the bracket

$$[x^if(t),x^jg(t)]=x^{i+j}(jf'(t)g(t)-if(t)g'(t))+i\delta_{i,-j}\,\mathrm{Res}_t\,t^{-1}f(t)g(t), \tag{2.11}$$

for $i,j\in\mathbb{Z}$ and $f(t),g(t)\in\mathbb{F}[t^{\pm1}]$, where the prime stands for the derivative $\frac{d}{dt}$, and $\mathrm{Res}_t\,f(t)$ stands for the *residue* of the Laurent polynomial $f(t)$, namely the coefficient of t^{-1} in $f(t)$. Denote by $M(\Lambda)$ the Verma $\mathcal{B}$-module with highest weight vector v_Λ, and by $L(\Lambda)=M(\Lambda)/M'$ the irreducible highest weight module of weight Λ, where M' is the maximal proper submodule of $M(\Lambda)$. Set

$$\mathcal{A}=\{a\in\mathcal{B}\,|\,av_\Lambda\in M'\}\quad\text{and}\quad\mathcal{P}=\mathcal{A}+\mathcal{B}_0. \tag{2.12}$$

Clearly, $\mathcal{B}_+\subset\mathcal{A}$, and $\mathcal{P}$ is a subalgebra of $\mathcal{B}$.

Theorem 1.1 will follow from the following proposition.

Proposition 2.1 *The following conditions are equivalent:*

(1) $M(\Lambda)$ *is reducible.*

(2) $\mathcal{P}_{-1}\neq\{0\}$.

(3) *There exists* $0\neq f(t)=\sum_{j=0}^m a_jt^j\in\mathbb{F}[t]$ *for some* $m\in\mathbb{N}$ *and* $a_j\in\mathbb{F}$, *such that* (*where* $a_{-k}=0$ *if* $k>0$ *or* $k<-m$)

$$\Lambda((t^kf(t))'-a_{-k}C)=0\ \textit{for all}\ k\in\mathbb{Z}. \tag{2.13}$$

(4) $\Sigma_\Lambda^{(j)}(z) := z\Delta_\Lambda^{(-j)}(z) - j\Delta_\Lambda^{(-j-1)}(z) - \frac{c}{j!}z^j$ *is a quasipolynomial and satisfies the same nontrivial linear differential equation with constant coefficients for all* $j \in \mathbb{Z}$.

(5) $L(\Lambda)$ *is quasifinite.*

Proof. We shall follow some arguments in [S1].

(1) $\Rightarrow$ (2): Assume $M' \neq 0$. Similar to the proof in [WS], we can prove that $\mathcal{P}$ is a parabolic subalgebra of $\mathcal{B}$, namely,

$$\mathcal{P} \supset \mathcal{B}_0 + \mathcal{B}_+ \neq \mathcal{P}, \quad \text{and} \quad \mathcal{P}_{-1} = \mathcal{P} \cap \mathcal{B}_{-1} \neq 0.$$

(2) $\Rightarrow$ (3): Let $f(t)$ be the monic ploynomial, called the *characteristic polynomial* of $\mathcal{P}$, with minimal degree such that $x^{-1}f(t) \in \mathcal{P}$ (cf. [S1, KL]). Set $a = x^{-1}f(t)$. Then from

$$[t^{-1}, x^{-1}tf(t)] = x^{-1}t^{-1}f(t), \qquad [t^k, x^{-1}f(t)] = -kx^{-1}t^{k-1}f(t) \quad \text{for} \quad k \in \mathbb{Z}, \tag{2.14}$$

we have $\mathcal{P}_{-1} = x^{-1}f(t)\mathbb{F}[t^{\pm 1}]$. Write $f(t) = \sum_{j=0}^m a_j t^j$ for some $m \in \mathbb{N}$ and $a_j \in \mathbb{F}$. From definition (2.12), we see M' is a proper submodule. We have $b \cdot av_\Lambda = 0$ for all $b \in \mathcal{B}_+$. In particular

$$[xt^k, x^{-1}f(t)]v_\Lambda = \Lambda(-(t^kf(t))' + a_{-k}C) = 0 \quad \text{for all} \quad k \in \mathbb{Z}.$$

(3) $\Rightarrow$ (4): Suppose we have (2.13). Using $e^{zt} = \sum_{i=0}^\infty \frac{z^i}{i!}t^i$ as a generating series of $\mathbb{F}[t]$, noting that $f(t)e^{zt} = f(\frac{\partial}{\partial z})e^{zt}$, by (2.13) and (2.11), we have (recall that the prime stands for $\frac{\partial}{\partial t}$, and the definition of $\Delta_\Lambda^{(j)}(z)$ in (1.3)),

$$\begin{aligned}
0 &= \Lambda((f(t)t^{-j}e^{zt})' - \mathrm{Res}_t\, t^{-j-1}f(t)e^{zt}C) \\
&= \Lambda((f(\frac{\partial}{\partial z})t^{-j}e^{zt})') - \mathrm{Res}_t\, t^{-j-1}f(\frac{\partial}{\partial z})e^{zt}c \\
&= \Lambda(-jf(\frac{\partial}{\partial z})t^{-j-1}e^{zt} + f(\frac{\partial}{\partial z})zt^{-j}e^{zt}) - f(\frac{d}{dz})\frac{z^j}{j!}c \\
&= -jf(\frac{\partial}{\partial z})\Lambda(t^{-j-1}e^{zt}) + f(\frac{\partial}{\partial z})z\Lambda(t^{-j}e^{zt}) - f(\frac{d}{dz})\frac{z^j}{j!} \\
&= -jf(\frac{d}{dz})\Delta_\Lambda^{(-j-1)}(z) + f(\frac{d}{dz})z\Delta_\Lambda^{(-j)}(z) - f(\frac{d}{dz})\frac{z^j}{j!}c \\
&= f(\frac{d}{dz})\Sigma_\Lambda^{(j)}(z) \quad \text{for } j \in \mathbb{Z}.
\end{aligned} \tag{2.15}$$

Namely, $\Sigma_\Lambda^{(j)}(z)$ is a quasipolynomial by the statement after (1.3).

(4) $\Rightarrow$ (5): Assume we have (2.15). It suffices to prove $\mathcal{P}_{-i}$ has finite codimension in $\mathcal{B}_{-i}$ for all $i > 0$. For this, we only need to prove $\mathcal{P}_{-i} \neq 0$ for all $i > 0$ (then we can proceed as in (2.14) to prove $\mathcal{P}_{-i}$ has finite codimension).

First from definition (2.12), $a \in \mathcal{P}_{-1}$ if and only if $b \cdot a \cdot v_\Lambda = 0$ for all $b \in \mathcal{B}_1$. Then from (2.15), one has

$$x^{-1}t^j f(t) \in \mathcal{P}_{-1} \quad \text{for all} \quad j \in \mathbb{Z}, \tag{2.16}$$

this is because $xt^k \cdot x^{-1}t^j f(t) \cdot v_\Lambda = 0$ for all $k \in \mathbb{Z}$, i.e., $\mathcal{B}_1 \cdot x^{-1}t^j f(t) \cdot v_\Lambda = 0$. In particular, from (2.16), one has $\mathcal{P}_{-1} \neq 0$. Assume that $\mathcal{P}_{-i} \neq 0$ and $0 \neq x^{-i}g(t) \in \mathcal{P}_{-i}$. Then

$$x^{-i-1}(-ijt^{j-1}f(t)g(t) - it^j f'(t)g(t) + t^j f(t)g'(t)) = [x^{-1}t^j f(t), x^{-i}g(t)] \in \mathcal{P}_{-i-1},$$

for all $j \in \mathbb{Z}$. In particular, $\mathcal{P}_{-i-1} \neq 0$.

(5) $\Rightarrow$ (1): It follows from (2.4). $\square$

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References

[B] R. Block, On torsion-free abelian groups and Lie algebras, *Proc. Amer. Math. Soc.* **9** (1958), 613–620.

[DZ] D. Dokovic, K. Zhao, Derivations, isomorphisms and cohomology of generalized Block algebras, *Algebra Colloq.* **3** (1996), 245–272.

[KL] V. Kac, J. Liberati, Unitary quasifinite representations of W_∞, *Lett. Math. Phys.* **53** (2000), 11–27.

[KR] V. Kac, A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Comm. Math. phys.* **157** (1993), 429–457.

[KWY] V. Kac, W. Wang, C.H. Yan, Quasifinite representations of classical Lie subalgebras of $\mathcal{W}_{1+\infty}$, *Adv. Math.* **139** (1998), 46–140.

[LZ] N. Lam, R.B. Zhang, Quasifinite modules for Lie superalgebras of infinite rank, *Tran. Amer. Math. Soc.* **358** (2005), 403–439.

[LW] S. Lin, Y. Wu, Representations of the quantized Weyl algebra associated to the quantum plane, *Algebra Colloquium* **12** (2005), 715–720.

[OZ] J. M. Osborn, K. Zhao, infinite-dimensional Lie algebras of generalized Block type, *Proc. Amer. Math. Soc.* **127** (1999), 1641–1650.

[S1] Y. Su, Derivations and structure of the Lie algebras of Xu type, *Manuscripta Math.* **105** (2001), 483–500.

[S2] Y. Su, Structure of Lie superalgebras of Block type related to locally finite derivations, *Comm. Algebra* **30** (2002), 3205–3226.

[S3] Y. Su, Quasifinite representations of a Lie algebra of Block type, *J. Algebra* **276** (2004), 117–128.

[S4] Y. Su, Quasifinite representations of a family of Lie algebras of Block type, *J. Pure Appl. Algebra* **192** (2004), 293–305.

[S5] Y. Su, Classification of quasifinite modules over the Lie algebras of Weyl type, *Adv. Math.* **174**(2003), 57–68.

[S6] Y. Su, Poisson brackets and structure of nongraded Hamiltonian Lie algebras related to locally-finite derivations, *Canad. J. Math.* **55** (2003), 856–896.

[SX1] Y. Su, X. Xu, Structures of divergence-free Lie algebras, *J. Algebra* **243** (2001), 557–595.

[SX2] Y. Su, X. Xu, Central simple Poisson algebras, *Science in China A* **47** (2004), 245–263.

[SZ] Y. Su, J. Zhou, Structure of the Lie algebras with a feature of Block algebras, *Comm. Algebra* **30** (2002), 3205–3226.

[WS] Y. Wu, Y. Su, Highest weight representations of a Lie algebra of Block type, preprint (arXiv: math.QA/0511733).

[X1] X. Xu, Generalizations of Block algebras, *Manuscripta Math.* **100** (1999), 489–518.

[X2] X. Xu, Quadratic conformal superalgebras, *J. Algebra* **231** (2000), 1–38.

[X3] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, *J. Algebra* **224** (2000), 23–58.

[ZM] L. Zhu, D. Meng, Structure of degenerate Block algebras, *Algebra Colloq.* **10** (2003), 53–62.